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# MAPPINGS ONTO METRIC SPACES

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We give characterizations of perfect images and open and compact images of spaces that can be mapped onto metrizable spaces by a mapping with fibers having a given property $\mathcal{P}$. We use these characterizations to obtain conditions which imply that such images can be mapped onto a metric space by a mapping with fibers satisfying $\mathcal{P}$. Such a treatment includes the investigation of spaces with a weaker metric topology [2, Ch. 5].

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weaker metric topology
perfect mappings
weaker developable topology
open and compact mappings

We investigate the class of spaces that can be mapped onto metric spaces by a mapping with fibers having a given property $\mathcal{P}$. We obtain characterizations of perfect images and open and compact images of elements of this class. We also give sufficient conditions for such images to belong to this class. Similar results can be obtained for spaces that can be mapped onto developable spaces. These results are of particular interest if $\mathcal{P}$ is a property of being a one-point set for then the spaces in question are spaces that have a weaker metrizable (developable) topology and their perfect images and open and compact images.

In the first section we discuss spaces that can be mapped onto metric spaces. Such spaces are precisely spaces that admit a continuous pseudometric. The following two results correspond to the two methods of constructing continuous pseudometrics [5, 5.4.H] (functionally open cover is a cover consisting of functionally open (= cozero) sets).

**0.1.** A space $X$ admits a (closed) mapping with fibers having a property $\mathcal{P}$ onto a metric space iff $X$ has a normal sequence of open covers $\{\mathcal{G}_n\}_{n\geq 1}$ such that $\bigcap_{n\geq 1} \mathrm{St}(x, \mathcal{G}_n) \in \mathcal{P}$ (and $\{\mathrm{St}(x, \mathcal{G}_n)\}_{n\geq 1}$ forms a base of neighbourhoods of this intersection) for $x \in X$.

**0.2.** Assume that $\mathcal{P}$ is hereditary with respect to closed subsets. A space $X$ admits a mapping with fibers having a property $\mathcal{P}$ onto a metric space iff $X$ has a sequence

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of ($\sigma$-) locally finite functionally open covers $\{\mathcal{U}_n\}_{n\geq 1}$ such that for each $x \in X$ there exists a sequence $\{U_n\}_{n\geq 1}$ such that $x \in U_n \in \mathcal{U}_n$ for $n \geq 1$ and closed subsets of $\bigcap_{n\geq 1} U_n$ have the property $\mathcal{P}$.

If we restrict ourselves to countable covers in 0.1 and 0.2, then we get characterizations of spaces that can be mapped onto separable metric spaces.

The usefulness of 0.1 is demonstrated in [7]. The way we shall use 0.2 can be illustrated by the following result (see [4], [10]).

**0.3.** Assume that $\mathcal{P}$ is hereditary with respect to closed subsets and invariant under open and perfect mappings. If $Y$ is an open and perfect image of a space $X$ which has a mapping with fibers having $\mathcal{P}$ onto a (separable) metric space, then $Y$ has such a mapping too.

To see that 0.3 holds, it is sufficient to observe that open and perfect mappings preserve the property of being a locally finite functionally open cover [5, 1.5.L and 3.10.11].

In the second section we discuss spaces that can be mapped onto developable spaces. This requires the change of the techniques. The notion of a neighbournet [8] turns out to be useful when dealing with such spaces.

We shall use the terminology and notation from [5]. By a mapping we always mean a continuous function. All spaces are assumed to be Hausdorff unless otherwise is stated.

Develpable spaces are spaces with a developmennt [5, p. 408] and Moore spaces are regular developable spaces. A space $X$ is said to be a $\sigma$-space if it has a $\sigma$-locally finite closed network [13]. Developable spaces are $\sigma$-spaces.

A cover $\mathcal{D}$ of $X$ is interior-preserving [8] if $\operatorname{Int}(\bigcap \mathcal{D}') = \bigcap\{\operatorname{Int} D: D \in \mathcal{D}'\}$ for all $\mathcal{D}' \subset \mathcal{D}$ and the space $X$ is orthocompact if each open cover of $X$ has an interior-preserving open refinement [6].

If $\mathcal{D}$ is a family of subsets of $X$ and $x \in X$, then $\mathcal{D}(x) = \{D \in \mathcal{D}: x \in D\}$.

If $\mathcal{M}$ is a class of spaces $\mathcal{P}(\mathcal{M})$ will denote the class of spaces that can be mapped onto a space from $\mathcal{M}$ by a mapping with fibers from $\mathcal{P}$.

When discussing a mapping defined on a space from $\mathcal{P}(\mathcal{M})$ we shall always assume that the image of a subset with the property $\mathcal{P}$ has this property. This assumption is obviously satisfied in the most interesting cases, when $\mathcal{P}$ is the property of being a one-point set, finite set or a compact space.

## 1. Mappings onto metric spaces

We shall give characterizations of perfect images and open and compact images of spaces from $\mathcal{P}$ (metric). In order to formulate these characterizations without any additional restrictions on $\mathcal{P}$, we have to introduce the notion of a sieve. If $\mathcal{P}$

is a hereditary property, then the characterizations can be formulated without using this notion.

**Definition 1.1** [3]. A sequence $\mathcal{D} = \{(\mathcal{D}_n, A_n, \pi_n)\}_{n\geq 1}$ will be called a *sieve* of $Y$ if each $\mathcal{D}_n = \{D(\alpha): \alpha \in A_n\}$ is a cover of $Y$ and $\pi_n : A_{n+1} \to A_n$ is such that for $\alpha \in A_n$ $D(\alpha) = \bigcup\{D(\alpha'): \pi_n(\alpha') = \alpha\}$. If each $\mathcal{D}_n$ is, as an indexed cover, locally finite and closed (point-finite and open), then $\mathcal{D}$ will be called a locally finite closed (point-finite open) sieve. A sequence $\{D(\alpha_n)\}_{n\geq 1}$, where $\alpha_n \in A_n$ and $\pi_n(\alpha_{n+1}) = \alpha_n$ for $n \geq 1$, will be called a *thread* of $\mathcal{D}$.

**Theorem 1.1.A.** *For any space $Y$ the following are equivalent*:

(1) *$Y$ is a perfect image of a space from $\mathcal{P}$(metric)*,

(2) *$Y$ has a locally finite closed sieve $\mathcal{F} = \{(\mathcal{F}_n, A_n, \pi_n)\}_{n\geq 1}$ such that the intersection of each thread of $\mathcal{F}$ has the property $\mathcal{P}$.*

*Furthermore, if $\mathcal{P}$ is hereditary with respect to closed sets, then* (1) *is equivalent to*:

(3) *$Y$ has a sequence $\{\mathcal{F}_n\}_{n\geq 1}$ of locally finite closed covers such that for each $y \in Y$ and $\{F_n\}_{n\geq 1}$ with $y \in F_n \in \mathcal{F}_n$ for $n \geq 1$, $\bigcap_{n\geq 1} F_n \in \mathcal{P}$.*

**Theorem 1.1.B.** *For any space $Y$ the following are equivalent*:

(1) *$Y$ is an open and compact image of a space from $\mathcal{P}$(metric)*,

(2) *$Y$ has a point-finite open sieve $\mathcal{V} = \{(\mathcal{V}_n, A_n, \pi_n)\}_{n\geq 1}$ such that the intersection of each thread of $\mathcal{V}$ has the property $\mathcal{P}$.*

*Furthermore, if $\mathcal{P}$ is a hereditary property, then* (1) *is equivalent to*:

(3) *$Y$ has a sequence $\{\mathcal{V}_n\}_{n\geq 1}$ of point-finite open covers such that for each $y \in Y$ and $\{V_n\}_{n\geq 1}$ with $y \in V_n \in \mathcal{V}_n$ for $n \geq 1$, $\bigcap_{n\geq 1} V_n \in \mathcal{P}$.*

We shall prove Theorem 1.1.A; the proof of B is similar.

**Proof of Theorem 1.1.A.** The implication (1)$\Rightarrow$(2) is obvious. To prove (2)$\Rightarrow$(1), consider $M = \varprojlim(A_n, \pi_n)$ with the Baire metric [5, 4.2.12] and put

$$X = \{(y, \{\alpha_n\}) \in Y \times M : y \in \bigcap_{n\geq 1} F(\alpha_n)\}.$$

Let $f$ and $p$ be the restrictions to $X$ of the projections of $Y \times M$ onto $Y$ and $M$ respectively. Clearly the fibers of $p$ have the property $\mathcal{P}$ and the fibers of $f$ are compact. Thus $X \in \mathcal{P}$(metric) and it is sufficient to prove that $f$ is a closed mapping.

Let $U$ be a neighbourhood of $f^{-1}(y)$ in $Y \times M$. From the compactness of $f^{-1}(y)$, it follows [5, 3.2.10] that for some $V$ open in $Y$ and $m \geq 1$,

$$f^{-1}(y) \subseteq V \times B(pf^{-1}(y), 1/m) \subseteq U.$$

Define $W = V \setminus \bigcup_{n\leq m} \bigcup \mathcal{F}_n \setminus \mathcal{F}_n(y)$. To show that $f^{-1}(W) \subseteq U$, take $y' \in W$ and $x' = (y', \{\alpha'_n\}) \in f^{-1}(y')$. Since $y' \in W \cap \bigcap_{n\geq 1} F(\alpha'_n)$, the definition of $W$ gives that $y \in \bigcap_{n\leq m} F(\alpha'_n)$ and, consequently, one can find a thread $\{F(\alpha_n)\}_{n\geq 1}$ of $\mathcal{F}$ such that $\alpha_n = \alpha'_n$ for $n \leq m$ and $y \in \bigcap_{n\geq 1} F(\alpha_n)$. Thus $\{\alpha'_n\} \in B(pf^{-1}(y), 1/m)$ and $x' \in U$.

The implication (3)$\Rightarrow$(2) is easy to see and (2)$\Rightarrow$(3) follows from the fact that for the locally finite closed sieve $\mathcal{F}=\{(\mathcal{F}_n, A_n, \pi_n)\}_{n\geq 1}$ of $Y$ and any sequence $\{F_n\}_{n\geq 1}$ with $F_n \in \mathcal{F}_n$ and $\bigcap_{n\geq 1} F_n \neq \emptyset$, there exists [5, 3.2.13] a thread $\{F(\alpha_n)\}_{n\geq 1}$ of $\mathcal{F}$ satisfying $\bigcap_{n\geq 1} F_n \subseteq \bigcap_{n\geq 1} F(\alpha_n)$. Thus (2) implies (3) if $\mathcal{P}$ is hereditary with respect to closed sets.

**Remark 1.1.** In order to obtain characterizations of images of spaces from $\mathcal{P}$(metric of weight $\leq \mathfrak{m}$) one has to assume that the cardinality of each $A_n$ in (2) or $\mathcal{F}_n(\mathcal{V}_n)$ in (3) is not greater than $\mathfrak{m}$. From the fact that each metric space of weight not greater than the continuum has a weaker separable metric topology [5, 4.4.C], it follows that $\mathcal{P}$(separable metric) is equal to $\mathcal{P}$(metric of weight $\leq \mathfrak{c}$).

If $\mathcal{P}$ is the property of being a one-point set, then the elements of $\mathcal{P}$(metric) are simply spaces with a weaker metric topology. Thus Theorems 1.1 (and Remark 1.1) yield.

**Corollary 1.1.A.** *A space $Y$ is a perfect image of a space with a weaker metric (separable metric) topology iff $Y$ has a sequence $\{\mathcal{F}_n\}_{n\geq 1}$ of (countable) locally finite closed covers such that $\bigcap_{n\geq 1} \mathrm{St}(y, \mathcal{F}_n)=\{y\}$ for $y \in Y$.*

**Corollary 1.1.B.** *A space $Y$ is an open and compact image of a space with a weaker metric (separable metric) topology iff $Y$ has a sequence of (countable) point-finite open covers $\{\mathcal{V}_n\}_{n\geq 1}$ such that $\bigcap_{n\geq 1} \mathrm{St}(y, \mathcal{V}_n)=\{y\}$ for $y \in Y$.*

The following example shows that one cannot claim that $\bigcap_{n\geq 1} \mathrm{St}(y, \mathcal{F}_n) \in \mathcal{P}$ in (3) of Theorem 1.1.A.

**Example 1.1.** Let $X = C_1 \cup C_2$ be the Alexandroff double circle [5, 3.1.26]. The natural projection is a two-to-one mapping of $X$ onto $C_1$. The mapping $f$ identifying $C_1$ to a point is a perfect mapping of $X$ onto $Y$ which is the Alexandroff compactification of the discrete space $C_2$. If $\mathcal{F}$ is a locally finite closed cover of $Y$ and $y$ is the accumulation point of $Y$, then $y \in \mathrm{Int}\, \mathrm{St}(y, \mathcal{F})$. Consequently, $Y \setminus \mathrm{St}(y, \mathcal{F})$ is finite and $\bigcap_{n\geq 1} \mathrm{St}(y, \mathcal{F}_n)$ is uncountable for any sequence $\{\mathcal{F}_n\}_{n\geq 1}$ of locally finite closed covers of $Y$.

Observe that $Y$ does not have any finite-to-one mapping onto a metric space.

The above example shows that $\mathcal{P}$(metric) is not closed under the action of perfect mappings in general. Some sufficient conditions for perfect images and open and compact images of spaces in $\mathcal{P}$(metric) to be in $\mathcal{P}$(metric) are given in the next two theorems.

**Theorem 1.2.A.** *Assume that $\mathcal{P}$ is hereditary with respect to closed sets. If a collectionwise normal and perfect space $Y$ is a perfect image of a space $X \in \mathcal{P}$(metric), then $Y \in \mathcal{P}$(metric).*

**Proof.** Let $\{\mathcal{F}_n\}_{n\geq 1}$ be a sequence of locally finite closed covers of $Y$ such that $\bigcap_{n\geq 1} F_n \in \mathcal{P}$ for any choice of $F_n \in \mathcal{F}_n(y)$. Since $Y$ is perfect and collectionwise normal, there exist, for $n, k \geq 1$, locally finite expansions $\mathcal{U}_{n,k} = \{U_{n,k}(F): F \in \mathcal{F}_n\}$ of $\mathcal{F}_n$ consisting of functionally open sets such that $F = \bigcap_{k\geq 1} U_{n,k}(F)$ for $F \in \mathcal{F}_n$ [5, 5.2.5, 5.5.18]. Thus, by virtue of 0.2, $Y \in \mathcal{P}$(metric).

**Theorem 1.2.B.** *Assume that $\mathcal{P}$ is hereditary with respect to closed sets. If a collectionwise normal space $Y$ is an open and compact image of a space $X \in \mathcal{P}$(metric), then $Y \in \mathcal{P}$(metric).*

**Proof.** Let $\mathcal{V} = \{(\mathcal{V}_n, A_n, \pi_n)\}_{n\geq 1}$ be a point-finite open sieve of $Y$ such that the intersection of each thread of $\mathcal{V}$ has the property $\mathcal{P}$. Since $Y$ is collectionwise normal, each $\mathcal{V}_n$ has a $\sigma$-locally finite functionally open refinement $\mathcal{U}_n$ [5, 5.3.3]. If $y \in U_n \in \mathcal{U}_n$ for $n \geq 1$, then, by virtue of [5, 3.2.13], there exists a thread $\{V(\alpha_n)\}_{n\geq 1}$ of $\mathcal{V}$ such that $\bigcap_{n\geq 1} U_n \subseteq \bigcap_{n\geq 1} V(\alpha_n)$. Thus each closed subset of $\bigcap_{n\geq 1} U_n$ has the property $\mathcal{P}$ and, from 0.2, it follows that $Y \in \mathcal{P}$(metric).

**Remark 1.2.** If $Y$ in Theorems 1.2 is an image of a space $X \in \mathcal{P}$(separable metric), then each $\mathcal{F}_n(A_n)$ is countable and the collectionwise normality of $Y$ can be replaced by normality. The assumption $X \in \mathcal{P}$(separable metric) is, by virtue of Remark 1.1, fulfilled if discrete collections of subsets of $Y$ are of cardinality not greater than continuum. Thus the assumption that $Y$ is collectionwise normal can be replaced by the assumption that $Y$ is normal and discrete collections of subsets of $Y$ are of cardinality not greater than continuum.

If $\mathcal{P}$ is the property of being a one-point set, then we obtain the following corollaries (the separable version for perfect mappings follows from [9]).

**Corollary 1.2.A.** *If a collectionwise normal (normal) and perfect space $Y$ is a perfect image of a space $X$ with a weaker (separable) metric topology, then $Y$ has a weaker (separable) metric topology.*

**Corollary 1.2.B.** *If a collectionwise normal (normal) space $Y$ is an open and compact image of a space $X$ with a weaker (separable) metric topology, then $Y$ has a weaker (separable) metric topology.*

Example 1.1 shows that the assumption that $Y$ is perfect in Theorem 1.2.A is essential. An example constructed in [14] (see [16]) shows that this assumption is also essential in Corollary 1.2.A.

An example showing that the assumption of normality of $Y$ is essential, is constructed in [17] (see [11] and [12]).

**Example 1.2.** [17, 3.5, 5.5, 5.7]. There exists a closed finite-to-one mapping $f: X \to Y$ of a metacompact Moore space $X$ (which is the Heath's $V$-space and is obtained by the strengthening of the topology of the plane) onto a metacompact Moore space $Y$ which does not have any weaker metrizable topology. Modifying [17, 3.5] one can prove that $Y$ does not even have any finite-to-one mapping onto a metric space.

Clearly, Example 1.2 shows that the normality of $Y$ is essential in 1.2.A. On the other hand, any metacompact Moore space is an open and compact image of a metrizable space [1], thus Example 1.2 shows that the normality of $Y$ is also essential in 1.2.B.

## 2. Mappings onto developable spaces

We have proved that perfect (open and compact) images of spaces from $\mathcal{P}$(metric) can be characterized in terms of sequences of locally finite closed (point-finite open) covers. In this section, we show that if a space $Y$ has such sequence satisfying an additional condition, then $Y \in \mathcal{P}$(developable $T_1$) (a strengthening of this condition indicated in Remark 2.1 gives another characterization of spaces in $\mathcal{P}$(metric)). Furthermore, we prove that if $Y$ is perfect, then any sequence of locally finite closed (point-finite open) covers of $Y$ can be modified so that it satisfies this condition. Therefore, if $Y$ is a perfect space which is a perfect (open and compact) image of a space $X \in \mathcal{P}$(metric), then $Y \in \mathcal{P}$(developable $T_1$). At last, we observe that if $Y$ is collectionwise normal and $Y \in \mathcal{P}$(developable $T_1$), then $Y \in \mathcal{P}$(metric).

Thus the results of this section can be considered as a deeper analysis of the results obtained in the first section.

We start with a characterization of spaces in $\mathcal{P}$(developable $T_1$).

**Theorem 2.1.A.** *The following conditions are equivalent for a space $X$:*

(1) $X \in \mathcal{P}$(*orthocompact developable* $T_1$),

(2) $X \in \mathcal{P}$(*developable* $T_1$),

(3) $X \in \mathcal{P}$($T_1$ *σ-space*),

(4) *$X$ has a sequence $\{\mathcal{E}_n\}_{n\geq 1}$ of locally finite closed covers satisfying*

(i) $\bigcap_{n\geq 1}\bigcap \mathcal{E}_n(x) \in \mathcal{P}$ *for* $x \in X$,

(ii) *for every $x \in X$ and $n \geq 1$ there exists a $k \geq 1$ such that*

$$\bigcap \mathcal{E}_k(x) \subseteq X\setminus\bigcup(\mathcal{E}_n\setminus\mathcal{E}_n(x)),$$

(5) *$X$ has a sequence $\{\mathcal{E}_n\}_{n\geq 1}$ of closure preserving closed covers satisfying* (i) *and* (ii).

(6) *$X$ has a sequence $\{\mathcal{U}_n\}_{n\geq 1}$ of interior preserving open covers satisfying*

(i') $\bigcap_{n\geq 1}\bigcap \mathcal{U}_n(x) \in \mathcal{P}$ *for* $x \in X$,

(ii') *for every $x \in X$ and $n \geq 1$ there exists a $k \geq 1$ such that*

$$X\setminus\bigcup(\mathcal{U}_k\setminus\mathcal{U}_k(x)) \subseteq \bigcap \mathcal{U}_n(x).$$

Before proving Theorem 2.1.A, we introduce some notation [8].

Any collection $\{R(x): x \in X\}$ of subsets of $X$ indexed by the points of $X$ can be associated with a binary relation $R$ in $X$ defined by $x' R x$ iff $x' \in R(x)$. This gives a natural one-to-one correspondence which explains the meaning of $\{R^{-1}(x): x \in X\}$ and $\{R \circ R'(x): x \in X\}$ for given collections (relations) $R$ and $R'$. In particular,

$$\begin{aligned} R \circ R^{-1}(X) &= \bigcup \{R(x'): x' \in R^{-1}(x)\} \\ &= \bigcup \{R(x'): x \in R(x')\} \\ &= \mathrm{St}(x, \mathcal{R}) \quad \text{where } \mathcal{R} = \{R(x): x \in X\}. \end{aligned}$$

If $\mathcal{R}$ is an arbitrary cover of $X$ and $R(x) = \bigcap \mathcal{R}(x)$, then $R$ is a transitive and reflexive relation and $R^{-1}(x) = X \setminus \bigcup (\mathcal{R} \setminus \mathcal{R}(x))$. The relation $R$ induces a new cover $\mathcal{R}' = \{R(x): x \in X\}$. It is easy to see that $\bigcap \mathcal{R}'(x) = R(x)$ and, consequently, $X \setminus \bigcup (\mathcal{R}' \setminus \mathcal{R}'(x)) = R^{-1}(x)$.

**Proof of Theorem 2.1.A.** The implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5)$ are obvious.

To prove $(5) \Rightarrow (6)$, take a sequence $\{\mathcal{E}_n\}_{n \geq 1}$ of covers of $X$ as in (5) and put

$$E_n(x) = \bigcap \mathcal{E}_n(x), \qquad U_n(x) = X \setminus \bigcup (\mathcal{E}_n \setminus \mathcal{E}_n(x)).$$

Let $\mathcal{U}_n = \{U_n(x): x \in X\}$. Clearly $\mathcal{U}_n$ is an open cover of $X$. From the remarks preceding the proof, it follows that

$$\bigcap \mathcal{U}_n(x) = U_n(x) = E_n^{-1}(x) \quad \text{and} \quad X \setminus \bigcup (\mathcal{U}_n \setminus \mathcal{U}_n(x)) = E_n(x).$$

Thus each $\mathcal{U}_n$ is interior preserving and (ii') is equivalent to (ii). From (ii'), it follows that $\bigcap_{n \geq 1} U_n^{-1}(x) \subseteq \bigcap_{n \geq 1} U_n(x)$. Therefore the relation $\bigcap_{n \geq 1} U_n$ is symmetric and (i') is equivalent to (i).

To prove $(6) \Rightarrow (1)$, take a sequence $\{\mathcal{U}_n\}_{n \geq 1}$ of covers of $X$ as in (6). Assume, without loss of the generality, that $\mathcal{U}_n = \{U_n(x): x \in X\}$, where $U_n(x) = \bigcap \mathcal{U}_n(x)$ and $U_{n+1}(x) \subseteq U_n(x)$ for $n \geq 1$ and $x \in X$.

Clearly, $R = \bigcap_{n \geq 1} U_n$ is a reflexive and transitive relation and (ii') implies that it is symmetric. Thus $R$ is an equivalence relation and (i') says that the equivalence classes of $R$ satisfy $\mathcal{P}$.

If $U_n(x) \in \mathcal{U}_n$ and $x' \in U_n(x)$, then $R(x') \subseteq U_n(x') \subseteq U_n(x)$, which shows that the elements of each $\mathcal{U}_n$ are saturated with respect to $R$.

If $x' \in R(x)$ and $n \geq 1$, then $x' \in U_n(x)$, which implies $U_n(x') \subseteq U_n(x)$ and shows, by the symmetry of $R$, that $U_n(x') = U_n(x)$.

Let $Z$ be the set of equivalence classes of $R$ and $p$ a natural function from $X$ onto $Z$. From the above observations, it follows that each $\mathcal{U}_n$ generates a cover $p(\mathcal{U}_n) = \{U_n(z): z \in Z\}$ of $Z$. Consider $Z$ with a topology generated by assuming that $\{U_n(z)\}_{n \geq 1}$ is a base of neighbourhoods of $z \in Z$. Clearly, $p$ is continuous and $\bigcap_{n \geq 1} U_n(z) = \{z\}$ implies that the fibers of $p$ have the property $\mathcal{P}$ and that $Z$ is a $T_1$ space. Since each $p(\mathcal{U}_n)$ is an interior preserving cover of $Z$, it remains to show that $\{p(\mathcal{U}_n)\}_{n \geq 1}$ is a development for $Z$ [6].

Take $z \in Z$ and an open set $U \subseteq Z$ containing $z$, then $z \in U_n(z) \subseteq U$ for some $n \geq 1$. Let $k \geq n$ satisfy (ii') of (6) for this $n$ and an $x \in p^{-1}(z)$, then

$$\begin{aligned} \mathrm{St}(z, p(\mathcal{U}_k)) &= p(\mathrm{St}(x, \mathcal{U}_k)) = p(U_k \circ U_k^{-1}(x)) \subseteq p(U_k \circ U_n(x)) \\ &\subseteq p(U_n \circ U_n(x)) = p(U_n(x)) = U_n(z). \end{aligned}$$

Observe that if the covers $\mathcal{U}_n$ in (6) are point-finite, then the covers $\{U_n(x): x \in X\}$, where $U_n(x) = \bigcap \mathcal{U}_n(x)$, are point-finite too. Thus the above proof gives:

**Theorem 2.1.B.** *The following conditions are equivalent for a space $X$:*

(1) *$X \in \mathcal{P}$(metacompact developable $T_1$),*

(2) *$X$ has a sequence $\{\mathcal{U}_n\}_{n \geq 1}$ of point-finite open covers satisfying* (i') *and* (ii').

**Remark 2.1.** The $k$ in (ii) and (ii') depends on $x$ and $n$. If one assumes that it depends on $n$ only, then one obtains a characterization of spaces in $\mathcal{P}$(zero-dimensional metric).

If (ii) in (4) (or (5)) is replaced by:

(iii) for every $x \in X$ and $n \geq 1$ there exists a $k \geq 1$ such that

$$\mathrm{St}(x, \mathcal{E}_k) \subseteq X \setminus \bigcup (\mathcal{E}_n \setminus \mathcal{E}_n(x)),$$

then one obtains a characterization of spaces in $\mathcal{P}$(metric) (see [5, 5.4.D]). The proof of Theorem 2.1.A shows that the similar replacement of (ii') by:

(iii') for every $x \in X$ and $n \geq 1$ there exists a $k \geq 1$ such that

$$\mathrm{St}(x, \mathcal{U}_k) \subseteq \bigcap \mathcal{U}_n(x),$$

gives another condition characterizing spaces in $\mathcal{P}$(developable $T_1$) and $\mathcal{P}$(metacompact developable $T_1$).

The space obtained by the identification of the corresponding accumulation points of the topological sum of two Michael's lines [16], [14] shows that the conditions (ii) and (ii') in the corresponding characterizations given in Theorems 2.1 are essential (for the property $\mathcal{P}$ of being a one-point set).

Conditions (ii) and (ii') can be destroyed by perfect (Example 1.1 or the example from [16], [14]) and open and compact mappings (Example 2.2 below). The next two lemmas show that these conditions can be reestablished if the image space is perfect.

As a consequence of these lemmas one can deduce that if $X$ is a perfect space, then (ii) in (4) of Theorem 2.1.A and (ii') in (2) of Theorem 2.1.B can be omitted (if $\mathcal{P}$ is hereditary with respect to closed sets).

**Lemma 2.2.A.** *If $\mathcal{F}$ is a closed locally finite cover of a perfect space $Y$, then there exists a countable family $\{\mathcal{F}_{i,j}\}_{i,j \geq 1}$ of locally finite closed covers of $Y$ satisfying:*

(i) *$\bigcap \mathcal{F}_{i,j}(y) \subseteq \bigcap \mathcal{F}(y)$ for $y \in Y$ and $i, j \geq 1$,*

(ii) *for each* $y \in Y$ *there exist* $i, j \geq 1$ *such that*

$$\bigcap \mathcal{F}_{i,j}(y) \subseteq Y \setminus \bigcup (\mathcal{F} \setminus \mathcal{F}(y)).$$

**Lemma 2.2.B.** *If* $\mathcal{V}$ *is a point-finite open cover of a perfect space* $Y$, *then there exists a countable family* $\{\mathcal{V}_{i,j}\}_{i,j\geq 1}$ *of point-finite open covers of* $Y$ *satisfying*

(i′) $\bigcap \mathcal{V}_{i,j}(y) \subseteq \bigcap \mathcal{V}(y)$ *for* $y \in Y$ *and* $i, j \geq 1$,

(ii′) *for each* $y \in Y$ *there exist* $i, j \geq 1$ *such that*

$$Y \setminus \bigcup (\mathcal{V}_{i,j} \setminus \mathcal{V}_{i,j}(y)) \subseteq \bigcap \mathcal{V}(y).$$

**Proof of Lemma 2.2.A.** Since $\mathcal{F}$ is closed and locally finite, it follows that the sets $Y_i = \{y \in Y : |\mathcal{F}(y)| \leq i\}$ are open. Thus $Y_i = \bigcup_{j\geq 1} F_{i,j}$, where $F_{i,j}$ are closed in $Y$. Put $\mathcal{F}_{i,j} = \mathcal{F} \cup \{F_{i,j}\}$. Obviously (i) is satisfied. To see (ii), take $y \in Y$ and let $i = |\mathcal{F}(y)|$. We have $y \in Y_i$ and, consequently, $y \in F_{i,j}$ for a certain $j \geq 1$.

Let us check that these $i$ and $j$ satisfy (ii) for $y$. If

$$y' \in \bigcap \mathcal{F}_{i,j}(y) = F_{i,j} \cap \bigcap \mathcal{F}(y) \subseteq Y_i \cap \bigcap \mathcal{F}(y),$$

then $|\mathcal{F}(y')| \leq i = |\mathcal{F}(y)|$ and $\mathcal{F}(y) \subseteq \mathcal{F}(y')$. Therefore $\mathcal{F}(y') = \mathcal{F}(y)$ and $y' \in Y \setminus \bigcup (\mathcal{F} \setminus \mathcal{F}(y))$.

The proof of Lemma 2.2.B is dual to the above proof. We shall indicate how to construct $\mathcal{V}_{i,j}$ from $\mathcal{V}$. Let $Y_i = \{y \in Y : |\mathcal{V}(y)| \geq i\}$. Clearly, $Y_i$ is open in $Y$ and, consequently, $Y_i = \bigcup_{j\geq 1} F_{i,j}$, where $F_{i,j}$ are closed in $Y$. Then $\mathcal{V}_{i,j} = \mathcal{V} \cup \{Y \setminus F_{i,j}\}$ satisfy (i′) and (ii′).

The following two theorems can be easily derived from Lemmas 2.2 and Theorems 2.1.

**Theorem 2.2.A.** *Assume that* $\mathcal{P}$ *is hereditary with respect to closed sets. If a perfect space* $Y$ *is a perfect image of a space* $X \in \mathcal{P}$*(developable* $T_1$*), then* $Y \in \mathcal{P}$*(developable* $T_1$*).*

**Theorem 2.2.B.** *Assume that* $\mathcal{P}$ *is hereditary with respect to closed sets. If a perfect space* $Y$ *is an open and compact image of a space* $X \in \mathcal{P}$*(metric), then* $Y \in \mathcal{P}$*(metacompact developable* $T_1$*).*

Theorems 2.2 are related to Theorems 1.2 by the following:

**Proposition 2.2.** *Assume that* $\mathcal{P}$ *is hereditary with respect to closed sets. If* $Y \in \mathcal{P}$*(developable* $T_1$*) is a collectionwise normal space, then* $Y \in \mathcal{P}$*(metric).*

**Proof.** Let $p : Y \to Z$ be a mapping of $Y$ onto a developable $T_1$ space $Z$ such that $p^{-1}(z) \in \mathcal{P}$ for $z \in Z$. Take a development $\{\mathcal{W}_n\}_{n\geq 1}$ of $Z$ and let $\mathcal{F}_n$ be a $\sigma$-discrete closed refinement of $\mathcal{W}_n$ for $n \geq 1$ [5, p. 409]. By the collectionwise normality of

$Y$, there exists a sequence $\{\mathcal{U}_n\}_{n\geq 1}$ of $\sigma$-discrete functionally open covers of $Y$ such that $\mathcal{U}_n = \{U_n(F): F \in \mathcal{F}_n\}$ refines $\{p^{-1}(W): W \in \mathcal{W}_n\}$ and $U_n(F) \supseteq p^{-1}(F)$ for $F \in \mathcal{F}_n$ and $n \geq 1$. From 0.2, it follows that $Y \in \mathcal{P}$(metric).

**Remark 2.2.** Using a method from [15, 2.5], one can prove that the assumption that $Y$ is collectionwise normal can be replaced by the assumption that $Y$ is normal and the discrete collections of subsets of $Y$ are of cardinality not greater than continuum (see Remark 1.2).

The comparison of Theorem 1.2.B with Theorem 2.2.B and Proposition 2.2 suggests that the assumption that $Y$ is a perfect space may be omitted in Theorem 2.2.B. The next example shows that this suggestion is wrong (for the property $\mathcal{P}$ of being a compact space).

**Example 2.2** [5, 2.3.36]. Let $Y = A(\aleph_1) \times A(\aleph_2) \setminus \{(a_1, a_2)\}$, where $A(\aleph_i)$ is the Alexandroff compactification of the discrete space of cardinality $\aleph_i$ and $a_i$ is the accumulation point of $A(\aleph_i)$.

It is easy to see that $Y$ is an open at most two-to-one image of a space $X$ that is a perfect preimage of a discrete space ($X$ is the topological sum of all the compact vertical and horizontal lines in $Y$).

From [5, 2.3.36], it follows that any mapping of $Y$ into the real line has a non-compact fiber and the only property of the line used in that proof is that its points are of the $G_\delta$-type. Thus the same proof shows that $Y \notin \mathcal{P}$(developable $T_1$) (for the property $\mathcal{P}$ of being a compact space).

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